1st Annual Lexington Mathematical Tournament
Team Round

Solutions

1 Potpourri

1. Answer: $\boxed{0}$
Solution: When we open the book, we will have an even-numbered page on the left and an odd-numbered page on the right. An even number and an odd number sum to an odd number, which cannot be divisible by 6. Thus, our probability is 0.

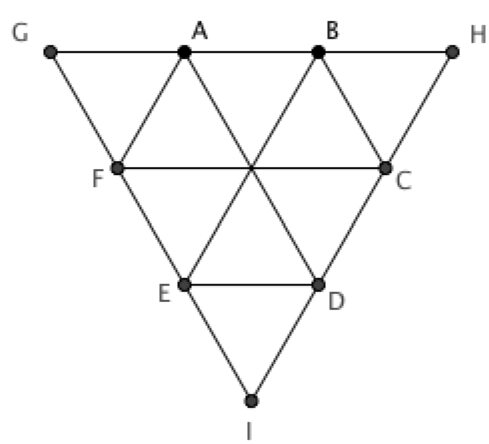
During the competition, we accepted a protest that claimed the probability was 1/1006, because opening to the last page gave 2010 on the left and a blank page on the right, which have a sum of 2010, divisible by 6. In this solution, there are 1006 possible places to open the book, and 1 gives a multiple of 6.

2. Answer: $\boxed{2}$
Solution: $128 = 2^7$ has 8 factors, namely $2^0 = 1, 2^1 = 2, \ldots, 2^7$, so $A = 8$.
$135 = 3^3 \cdot 5$, so $B = 3 + 5 = 8$.
We have $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243$. We see that the units' digits cycle by 4: 3, 9, , 1, 3, 9, 7, 1, etc. This is because obtaining the next power of 3 is the result of multiplying the previous one by 3, and the units' digit only depends on the previous units' digit. Because $81 = 4 \cdot 20 + 1$, the 81st term in this sequence is $B = 1$ (we go through 20 cycles of 4, then the 81st term).
The number of zeroes at the end of an integer is equivalent to the number of factors of $10 = 2 \cdot 5$ the integer has. We see that we can pull out a $2^3 \cdot 5^3 = 10^3$ from the prime factorization, and nothing more because we run out of factors of 5. Thus, $D = 3$.
$E = 9 \cdot 111 = 27 \cdot 37$, and 37 is prime, so $E = 37$.

Now, $\sqrt[3]{\sqrt{A+B} + \sqrt[3]{D^C + E}} = \sqrt[3]{\sqrt{16} + \sqrt[3]{64}} = \sqrt[3]{8} = 2$.

3. Answer: $\boxed{13}$
Solution: The root mean square is $\sqrt{\dfrac{17^2 + 7^2}{2}} = \sqrt{169} = 13$.

4. Answer: $\boxed{3/2}$



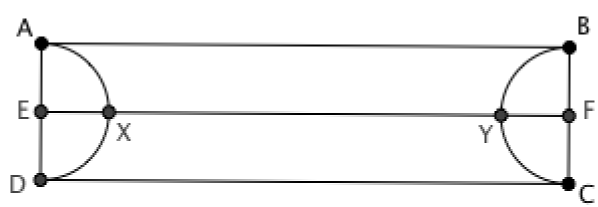
Solution: Notice that since $\angle BAF = 120°$, we have $\angle GAF = 180° - 120° = 60°$. Similarly, $\angle GFA = 60°$, so triangle GAF is equilateral. Similarly, BCH and IED are equilateral. Furthermore, the side lengths of the equilateral triangles are the same of the side lengths of the hexagon, since they share sides FA, BC, and DE. Now, we can draw diagonals AD, BE, CF to make six equilateral triangles in the interior of the hexagon. Each of these triangles has area $1/6$, since the area of the hexagon is 1. The area of the large polygon, triangle GHI, consists of 9 of these triangles, so the area we want is $9/6 = 3/2$.

5. Answer: $\boxed{-2}$
Solution: Note that $\sqrt{2}$ is between 1 and 2 and 2.5 is between 2 and 3. Thus, $\lfloor -2.5 \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor -\sqrt{2} \rfloor + \lfloor 2.5 \rfloor = (-3) + 1 + (-2) + 2 = -2$.

6. Answer: $\boxed{3}$
Solution: The 3rd largest number, the median, is 8, and the unique mode is 9. Clearly, there must be more than one 9, or else every number in the set would be a mode, and it would not be unique. However, there cannot be more than two, because the 3rd largest number is 8 (and the 9's can at best be the 2nd largest and largest). Thus, there are exactly two 9's, and our set consists of 8, 9, 9, and two positive integers smaller than 8. The third condition, that the mean of the integers is 7, means that the sum of the integers is 35. The sum of 8, 9, 9 is 26, so the sum of the last two integers is 9. There are three possible pairs of the two smallest integers, namely $(2, 7), (3, 6), (4, 5)$ (note that $(1, 8)$ fails because 9 is the unique mode). This gives us 3 sets in total.

7. Answer: $\boxed{81}$
If x is a three digit integer, it is between 100 and 999 inclusive, and $x + 1$ is between 101 and 1000 inclusive. We can simply count the number of multiples of 11 between 101 and 1000, inclusive, because each gives a unique corresponding x. $101/11$ is between 9 and 10, and $1000/11$ is between 90 and 91, so our multiples of 11 will go from $10 \cdot 11$ to $90 \cdot 11$, inclusive. This is equivalent to counting the number of integers between 10 and 90, inclusive, which is $90 - 10 + 1 = 81$.

8. Answer: $\boxed{5\pi/2}$



Solution: Let the centers of the semicircles be E, F. Since $AD = BC = 10$, $AE = BF = 5$, and the areas of the semicircles are both $\frac{1}{2}(5)^2\pi$, making the total area of the semicircles 25π. The total area inside the rectangle, minus the area of the semicircles, is 100, so the total area of the rectangle is $100 + 25\pi$. The minimum distance XY will occur when X and Y are farthest away from the sides AD and BC, respectively, that is, when on the line EF across the middle of $ABCD$. We have $XY = EF - EX - FY = EF - 10$. We know the area of the rectangle is $25\pi + 100$, and that $AD = 10$, so $EF = AB = \dfrac{25\pi + 100}{10} = \dfrac{5\pi}{2} + 10$. Thus, $XY = \dfrac{5\pi}{2}$.

9. Answer: $\boxed{46}$
Solution: There are $\dbinom{8}{3} = 56$ sets of three points, but not all sets form triangles, as some three sets consist of three collinear points (points on the same line). We know that if some three points

lie on the same line, they lie on l, and l contains 5 points. Thus, there are $\binom{5}{3} = 10$ sets of three points that lie on a line, leaving $56 - 10 = 46$ triangles.

10. Answer: $\boxed{\text{DHCAIBFEG}}$

Solution: Each book is only taken out and put in once. Thus, for each book in its ending position, the book was put in right when the book originally in that position was taken out. In other words, a book in some position was taken out right before the book originally in that position. For example, book C ends in the left-most position, which was originally occupied by A. Because A had to be taken out to put C in its place, and C was taken out right before it was placed in A's slot, A was taken out right after C.

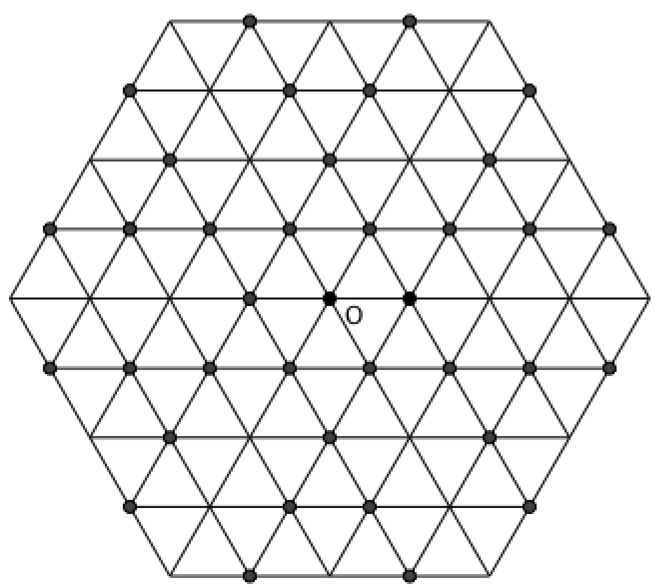
Now, we work backwards. Since Carl is holding book G at the end, the book in G's original position was the book Carl had in his hand before removing G; this is E. Before taking out E, Carl took out F, since F is in E's old position. Before F, he gook out B. Before B was I, before I was A, before A was C, before C was H, before H was D, and before D was J. Since J is the book in Carl's hand at the beginning, we have gone through the whole cycle. To get the answer, we now go backwards from backwards (which is forwards), to extract our answer: DHCAIBFEG.

During the competition, the problem mistakenly asked for a ten-letter string instead of a nine-letter string, even though Carl never "takes out" book J. As a result, we accepted both the above answer and the answer JDHCAIBFEG.

2 The Triangular Lattice

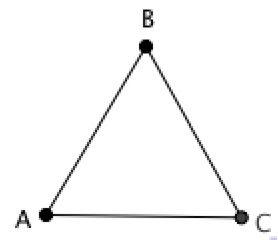
1. Solution: This problem is just a test of understanding of the relevant definitions. The answers are C, O, B, D, B, A, in that order. For (ii), note that the associated path is just one around an equilateral triangle of side length 2010, that returns to the origin at the end.

2.



The answer is 36, with visible points bolded in the diagram above (excluding O).

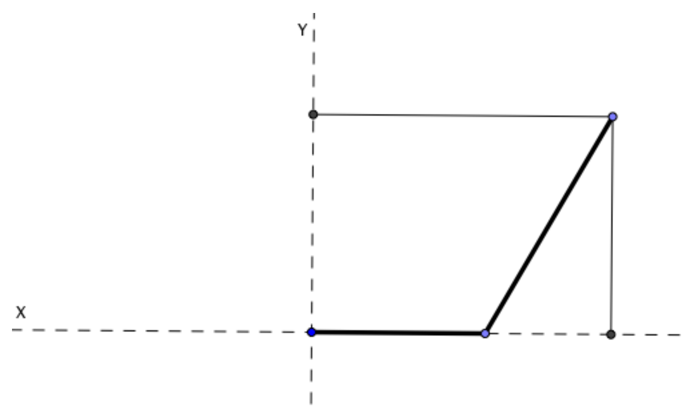
3.



Solution: Say $A = (a-1, b, c-1)$. Walking to $(a-1, b+1, c-1)$ takes us one step northeast, to B. Walking to (a, b, c) takes us one step east, then one step northwest, to C, then to B. Thus, we end up at the same point, so $(a, b, c) = (a-1, b+1, c-1)$.

4. Solution: If $c > 0$, we have $(a, b, c) = (a-1, b+1, c-1) = (a-2, b+2, c-2) = \cdots = (a-c, b+c, c-c) = (a-c, b+c, 0)$, since the integer that we subtract from a and c will grow $1, 2, \ldots, c$. If, on the other hand, $c < 0$, we instead start from $(a-c, b+c, 0)$, and apply the same 'operation', to get $(a-c, b+c, 0) = (a-c-1, b+c+1, -1) = (a-c-2, b+c+2, -2) = \cdots$. The sequence $0, -1, -2, \ldots, c$ will eventually hit c since $c < 0$, so we get that this is equal to $(a-c-(-c), b+c+(-c), 0-(-c)) = (a, b, c)$, so the claim is still true when $c < 0$.

5.



Solution: By problem 4, we have $(a, b, c) = (a-c, b+c, 0)$. Thus, we can take the path to our point that moves $a-c$ units east, then $b+c$ units in the $60°$ direction. To get the rectangular coordinates, we have a 30-60-90 triangle on the right with hypotenuse $b+c$, and the total distance that we move in the x-direction is the short leg of the triangle, which has length $(b+c)/2$. The total distance that we move in the y-direction has length $(b+c)\sqrt{3}/2$. Thus, we will move this distance in the y-direction and $(b+c)/2+a-c$ units in the x-direction, to get coordinates of $(a+\frac{b}{2}-\frac{c}{2}, \frac{(b+c)\sqrt{3}}{2})$.

Note: Although we didn't require the explanation for this, the same ideas also work if $a-c < 0$ or $b+c < 0$.

6. Solution: To get the distance between these two points, we first convert them to rectangular coordinates, which are $(a+\frac{b}{2}-\frac{c}{2}, \frac{(b+c)\sqrt{3}}{2})$, $(d+\frac{e}{2}-\frac{f}{2}, \frac{(e+f)\sqrt{3}}{2})$. From here, we can apply the distance formula in rectangular coordinates, to get a distance of

$$\sqrt{(a-d+\frac{b}{2}-\frac{c}{2}-\frac{e}{2}+\frac{f}{2})^2+(\frac{\sqrt{3}}{2}(e+f-b-c))^2};$$

$$= \sqrt{((a-d) + \frac{(b-e)-(c-f)}{2})^2 + \frac{3}{4}((f-c)-(b-e))^2};$$
$$= \sqrt{(a-d)^2 + (b-e)^2 + (c-f)^2 + (a-d)(b-e) + (b-e)(c-f) - (a-d)(c-f)}.$$

Note: We didn't expect a simplified expression like the one above, the top one would have been fine for full credit. The simplified expression, however, makes it easier to compute distances, as would be helpful in the next problem.

7. The triangle is equilateral. We will compute the lengths of the sides using the distance formula in part (f).
$(-1,4,-2),(-3,1,0)$: $\sqrt{2^2+3^2+(-2)^2+(2)(3)+(3)(-2)-(2)(-2)} = \sqrt{21}$;
$(-3,1,0),(0,-1,-2)$: $\sqrt{(-3)^2+2^2+2^2+(-3)(2)+(2)(2)-(2)(-3)} = \sqrt{21}$;
$(0,-1,-2),(-1,4,-2)$: $\sqrt{1^2+(-5)^2+0^2+(1)(-5)+(-5)(0)+(0)(1)} = \sqrt{21}$.

Thus, we get an equilateral triangle with side length $\sqrt{21}$.

Note: Can you find a faster solution, with no computation at all? (Hint: draw a picture, and look for congruent triangles)